Electronic Journal of Qualitative Theory of Differential Equations
Spec. Ed. I, 2009 No. **27**, 1–20; **http://www.math.u-szeged.hu/ejqtde/**

BIFURCATION ANALYSIS OF A KALDOR-KALECKI MODEL OF BUSINESS CYCLE WITH TIME DELAY

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

In this paper, we investigate a Kaldor-Kalecki model of business cycle with delay in both the gross product and the capital stock. Stability analysis for the equilibrium point is carried out. We show that Hopf bifurcation occurs and periodic solutions emerge as the delay crosses some critical values. By deriving the normal forms for the system, the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are established. Examples are presented to confirm our results.

Key words and phrases: Kaldor-Kalecki model of business cycle, Hopf bifurcation, periodic solutions, stability.
AMS (MOS) subject classifications: 34K18

1 Introduction

In this paper, we study the Kaldor-Kalecki model of business cycle with delay of the following form:

$$\begin{cases} \frac{dY(t)}{dt} = \alpha[I(Y(t), K(t)) - S(Y(t), K(t))], \\ \frac{dK(t)}{dt} = I(Y(t-\tau), K(t-\tau)) - qK(t), \end{cases} \tag{1}$$

where Y is the gross product, K is the capital stock, $\alpha > 0$ is the adjustment coefficient in the goods market, $q \in (0, 1)$ is the depreciation rate of capital stock, $I(Y, K)$ and $S(Y, K)$ are investment and saving functions, and $\tau \geq 0$ is a time lag representing delay for the investment due to the past investment decision.

A business model in this line was first proposed by Kalecki [11], in which the idea of a delay of the implementation of a business decision was introduced. Later on, Kalecki [12] and Kaldor [10] proposed and studied business models using ordinary differential equations and nonlinear investment and saving functions. They showed that periodic solutions exist under the assumption of nonlinearity. Similar models were also analyzed by several authors and the existence of limit cycles were established due to the nonlinearity, see [4, 7, 23]. Krawiec and Szydlowski [14, 15, 16] combined the two basic models of Kaldor's and Kalecki's and proposed the following Kaldor-Kalecki model of business cycle:

$$\begin{cases} \frac{dY(t)}{dt} = \alpha[I(Y(t), K(t)) - S(Y(t), K(t))], \\ \frac{dK(t)}{dt} = I(Y(t-\tau), K(t)) - qK(t). \end{cases}$$

This model has been studied intensively since its introduction, see [17, 19, 20, 21, 22, 24]. It is argued that a more reasonable model should include delays in both the gross product and capital stock, because the change in the capital stock is also caused by the past investment decisions [17]. Adding a delay to capital stock K leads to System (1).

As in [14], also see [1, 2, 22], using the following saving and investment functions S and I, respectively,

$$S(Y, K) = \gamma Y, \ I(Y, K) = I(Y) - \beta K$$

where $\beta > 0$ and $\gamma \in (0, 1)$ are constants, System (1) becomes the following system:

$$\begin{cases} \frac{dY(t)}{dt} = \alpha[I(Y(t)) - \beta K(t) - \gamma Y(t)], \\ \frac{dK(t)}{dt} = I(Y(t-\tau)) - \beta K(t-\tau) - qK(t). \end{cases} \tag{2}$$

Kaddar and Talibi Alaoui [9] studied System (2). They gave a condition for the characteristic equation of the linearized system to have a pair of purely imaginary roots and showed that the Hopf bifurcation may occur as the delay τ passes some critical values. However, they did not give the stability of the periodic solution and the direction of the Hopf bifurcation.

In this paper, we first give a more detailed discussion of the distribution of the eigenvalues of the linearized system of (2). So local stability of the equilibrium point is established. Conditions are found under which the Hopf bifurcation occurs and periodic solutions emerge as the delay crosses some critical values. By deriving the normal forms for System (2) using the normal form theory developed by Faria and Magalhães [5, 6], the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are established. Finally, some examples are presented to illustrate our theoretical results.

2 Distribution of Eigenvalues

Throughout the rest of this paper, we assume that $I(s)$ is a nonlinear function, C^3, and that System (2) has an isolated equilibrium point (Y^*, K^*). Let $I^* = I(Y^*)$, $u_1 = Y - Y^*, u_2 = K - K^*$, and $i(s) = I(s + Y^*) - I^*$. Then System (2) can be transformed into

$$\begin{cases} \frac{du_1(t)}{dt} = \alpha[i(u_1(t)) - \beta u_2(t) - \gamma u_1(t)], \\ \frac{du_2(t)}{dt} = i(u_1(t-\tau)) - \beta u_2(t-\tau) - q u_2(t). \end{cases} \tag{3}$$

Let the Taylor expansion of i at 0 be

$$i(u) = ku + i^{(2)}u^2 + i^{(3)}u^3 + O(|u|^4)$$

where

$$k = i'(0) = I'(Y^*),\ i^{(2)} = \frac{1}{2}i''(0) = \frac{1}{2}I''(Y^*),\ i^{(3)} = \frac{1}{3!}i'''(0) = \frac{1}{3!}I'''(Y^*).$$

The linear part of System (3) at $(0, 0)$ is

$$\begin{cases} \frac{du_1(t)}{dt} = \alpha[(k-\gamma)u_1(t) - \beta u_2(t)], \\ \frac{du_2(t)}{dt} = ku_1(t-\tau) - \beta u_2(t-\tau) - q u_2(t), \end{cases} \tag{4}$$

and its corresponding characteristic equation is

$$\lambda^2 + [q - \alpha(k-\gamma)]\lambda - \alpha q(k-\gamma) + (\beta\lambda + \alpha\beta\gamma)e^{-\lambda\tau} = 0. \tag{5}$$

For $\tau = 0$, Equation (5) becomes

$$\lambda^2 + [q + \beta - \alpha(k-\gamma)]\lambda - \alpha q(k-\gamma) + \alpha\beta\gamma = 0. \tag{6}$$

Define

$$k_1 = \frac{\beta\gamma}{q} + \gamma, \qquad k_2 = \frac{q+\beta}{\alpha} + \gamma,$$

and for the rest of the paper, we always assume $k_1 \leq k_2$. For the case that $k_1 > k_2$, the discussion can be carried out similarly.

Theorem 2.1. *Let $\tau = 0$. If $k < k_1$, all roots of Equation (6) have negative real parts, and hence (Y^*, K^*) is asymptotically stable. If $k > k_1$, Equation (6) has a positive root and a negative root, and hence (Y^*, K^*) is unstable.*

Now assume $\tau > 0$. Let ωi $(\omega > 0)$ be a purely imaginary root of Equation (5). After plugging it into Equation (5) and separating the real and imaginary parts, we have

$$\begin{aligned} \omega^2 + \alpha q(k-\gamma) &= \alpha\beta\gamma\cos(\omega\tau) + \beta\omega\sin(\omega\tau), \\ [q - \alpha(k-\gamma)]\omega &= \alpha\beta\gamma\sin(\omega\tau) - \beta\omega\cos(\omega\tau). \end{aligned} \tag{7}$$

Adding squares of two equations yields

$$\omega^4 + [q^2 - \beta^2 + \alpha^2(k-\gamma)^2]\omega^2 + \alpha^2 q^2 (k-\gamma)^2 - \alpha^2\beta^2\gamma^2 = 0. \tag{8}$$

Let

$$\begin{aligned} A &= q^2 - \beta^2 + \alpha^2(k-\gamma)^2, \\ B &= \alpha^2 q^2 (k-\gamma)^2 - \alpha^2\beta^2\gamma^2. \end{aligned}$$

If $A \geq 0$ and $B \geq 0$, Equation (8) has no positive roots. If $B < 0$, Equation (8) has a unique positive root

$$\omega_+ = \sqrt{\frac{-A + \sqrt{A^2 - 4B}}{2}}.$$

If $A < 0,\ B > 0$, and $A^2 - 4B > 0$, Equation (8) has two positive roots

$$\omega_\pm = \sqrt{\frac{-A \pm \sqrt{A^2 - 4B}}{2}}.$$

Solving Equation (7) for $\sin(\omega\tau)$ and $\cos(\omega\tau)$ yields

$$\begin{aligned} \sin(\omega\tau) &= \frac{\omega^3 + [\alpha q k - \alpha^2\gamma(k-\gamma)]\omega}{\alpha^2\beta\gamma^2 + \beta\omega^2}, \\ \cos(\omega\tau) &= \frac{\alpha^2 q\gamma(k-\gamma) + (\alpha k - q)\omega^2}{\alpha^2\beta\gamma^2 + \beta\omega^2}. \end{aligned}$$

Define

$$\begin{aligned} l_1^\pm &= \frac{\omega_\pm^3 + [\alpha q k - \alpha^2\gamma(k-\gamma)]\omega_\pm}{\alpha^2\beta\gamma^2 + \beta\omega_\pm^2}, \\ l_2^\pm &= \frac{\alpha^2 q\gamma(k-\gamma) + (\alpha k - q)\omega_\pm^2}{\alpha^2\beta\gamma^2 + \beta\omega_\pm^2}. \end{aligned}$$

We, thus, have the following result.

Lemma 2.1. *Let $\omega_\pm$ and $l_i^\pm$ $(i = 1, 2)$ be defined above.*

(i) *If $B < 0$, then there exists a sequence of positive numbers $\{\tau_j^+\}_{j=0}^\infty$ such that $\tau_0^+ < \tau_1^+ < \tau_2^+ < \cdots < \tau_j^+ < \cdots$, and Equation (5) has a pair of purely imaginary roots $\pm i\omega_+$ when $\tau = \tau_j^+$.*

(ii) *If* $A < 0,\ B > 0,$ *and* $A^2 - 4B > 0$, *then there exist two sequences of positive numbers* $\{\tau_j^+\}_{j=0}^{\infty}$ *and* $\{\tau_j^-\}_{j=0}^{\infty}$ *such that* $\tau_0^+ < \tau_1^+ < \tau_2^+ < \cdots < \tau_j^+ < \cdots$, $\tau_0^- < \tau_1^- < \tau_2^- < \cdots < \tau_j^- < \cdots$, *and Equation (5) has a pair of purely imaginary roots* $\pm i\omega_\pm$ *when* $\tau = \tau_j^\pm$.

Here $\tau_j^\pm$ $(j = 0, 1, 2, \cdots)$ *are defined below*

$$\tau_j^\pm = \frac{1}{\omega_\pm}\begin{cases} \arccos l_2^\pm + 2j\pi, & \text{if } l_1^\pm > 0, \\ 2\pi - \arccos l_2^\pm + 2j\pi, & \text{if } l_1^\pm < 0. \end{cases}$$

Define $\lambda(\tau) = \sigma(\tau) + i\omega(\tau)$ to be the root of Equation (5) such that $\sigma(\tau_j^\pm) = 0$ and $\omega(\tau_j^\pm) = \omega_\pm$, respectively.

Lemma 2.2. *Let* $\sigma(\tau)$ *and* $\tau_j^\pm$ *be defined above. Then*

$$\sigma'(\tau_j^+) > 0, \qquad \sigma'(\tau_j^-) < 0.$$

Proof. Differentiate Equation (5) with respect to τ yields

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{[2\lambda + q - \alpha(k-\gamma)]e^{\lambda\tau} + \beta}{\lambda\beta(\lambda + \alpha\gamma)} - \frac{\tau}{\lambda}$$

and a calculation gives

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_j^\pm} = \frac{2\omega_\pm^2 + \alpha^2(k-\gamma)^2 + q^2 - \beta^2}{\beta^2(\alpha^2\gamma^2 + \omega_\pm^2)} = \frac{2\omega_\pm^2 + A}{\beta^2(\alpha^2\gamma^2 + \omega_\pm^2)}$$

which gives

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_j^\pm} = \frac{\pm\sqrt{A^2 - 4B}}{\beta^2(\alpha^2\gamma^2 + \omega_\pm^2)},$$

completing the proof. □

To discuss the distribution of the roots of Equation (5), we will need the following lemma due to Ruan and Wei [18].

Lemma 2.3. *Consider the exponential polynomial*

$$P(\lambda, e^{-\lambda\tau}) = p(\lambda) + q(\lambda)e^{-\lambda\tau}$$

where p, q *are real polynomials such that* $\deg(q) < \deg(p)$ *and* $\tau \geq 0$. *As* τ *varies, the total number of zeros of* $P(\lambda, e^{-\lambda\tau})$ *on the open right half-plane can change only if a zero appears on or crosses the imaginary axis.*

Now we turn our attention to the relationship between A, B and our system parameters. We look at the following two cases.

Case I. $\beta \leq q$. In this case, $A \geq 0$.

1. $B \geq 0 \Longleftrightarrow k \geq k_1$ or $k \leq -\beta\gamma/q + \gamma$;

2. $B < 0 \Longleftrightarrow |k - \gamma| < \beta\gamma/q$.

Case II. $\beta > q$. In this case,

1. $A \geq 0,\ B \geq 0 \Longleftrightarrow k \geq \max\{\sqrt{\beta^2 - q^2}/\alpha + \gamma, k_1\}$ or $k \leq \min\{-\sqrt{\beta^2 - q^2}/\alpha + \gamma, -\beta\gamma/q + \gamma\}$;

2. $B < 0 \Longleftrightarrow |k - \gamma| < \beta\gamma/q$;

3. $A < 0,\ B > 0 \Longleftrightarrow \beta\gamma/q < |k - \gamma| < \sqrt{\beta^2 - q^2}/\alpha$.

The discussions above, Theorem 2.1 and Lemmas 2.1, 2.2 and 2.3 imply the following Lemma 2.4.

Lemma 2.4. *Assume $\beta \leq q$. Let τ_j^+ be defined in Lemma 2.1. Then we have*

(i) *if $B \geq 0$, then all roots of Equation (5) have negative real parts when $k < -\beta\gamma/q + \gamma$ and Equation (5) has roots with negative real parts and roots with positive real parts when $k > k_1$;*

(ii) *if $B < 0$, or $|k - \gamma| < \beta\gamma/q$, all roots of Equation (5) have negative real parts for all $\tau \in [0, \tau_0^+)$; Equation (5) has a pair of purely imaginary roots $\pm i\omega_+$ and all other roots have negative real parts when $\tau = \tau_0^+$; it has $2(j+1)$ roots with positive real parts and all other roots have negative real parts when $\tau \in (\tau_j^+, \tau_{j+1}^+),\ j = 0, 1, 2, \cdots$.*

Lemma 2.5. *Assume $\beta > q$. Let $\tau_j^\pm$ be defined in Lemma 2.1. Then we have*

(i) *if $A \geq 0,\ B \geq 0$, then all roots of Equation (5) have negative real parts when $k < \min\{-\sqrt{\beta^2 - q^2}/\alpha + \gamma, -\beta\gamma/q + \gamma\}$, and Equation (5) has roots with negative real parts and roots with positive real parts when $k > \max\{\sqrt{\beta^2 - q^2}/\alpha + \gamma, k_1\}$;*

(ii) *if $B < 0$, or if $|k - \gamma| < \beta\gamma/q$, all roots of Equation (5) have negative real parts for all $\tau \in [0, \tau_0^+)$; Equation (5) has a pair of purely imaginary roots $\pm i\omega_+$ and all other roots have negative real parts when $\tau = \tau_0^+$; it has $2(j+1)$ roots with positive real parts and all other roots have negative real parts when $\tau \in (\tau_j^+, \tau_{j+1}^+),\ j = 0, 1, 2, \cdots$.*

(iii) *if $A < 0,\ B > 0$ and $A^2 - 4B > 0$, then we have $\beta\gamma/q < |k - \gamma| < \sqrt{\beta^2 - q^2}/\alpha$ and $A^2 - 4B > 0$. Assume that $A^2 - 4B > 0$. If $-\sqrt{\beta^2 - q^2}/\alpha + \gamma < k < -\beta\gamma/q + \gamma$, all roots of Equation (5) have negative real parts for all $\tau \in [0, \tau_0^+)$, Equation (5) has roots with positive real parts when $\tau \in (\tau_0^+, \tau_m^-)$ where m is the smallest positive integer such that $\tau_m^- > \tau_0^+$, it has a pair of purely imaginary roots $\pm i\omega_+$ and all other roots have negative real parts when $\tau = \tau_0^+$. if $\tau_m^- < \tau_1^+$, Equation*

(5) has two roots with positive real parts and all other roots have negative real parts when $\tau \in (\tau_0^+, \tau_m^-)$ and all roots of Equation (5) have negative real parts when $\tau \in (\tau_m^-, \tau_1^+)$. If $k_1 < k < \sqrt{\beta^2 - q^2}/\alpha + \gamma$, Equation (5) has roots with negative real parts and roots with positive real parts.

The following Hopf bifurcation theorems follow immediately.

Theorem 2.2. *Assume $\beta \leq q$. Let τ_j^+ be defined in Lemma 2.1. Then we have*

(i) *the equilibrium point (Y^*, K^*) is asymptotically stable for all $\tau \geq 0$ when $k < -\beta\gamma/q + \gamma$ and it is unstable for all $\tau \geq 0$ when $k > k_1$;*

(ii) *the equilibrium point (Y^*, K^*) is asymptotically stable for all $\tau \in [0, \tau_0^+)$ and unstable for all $\tau > \tau_0^+$ when $|k - \gamma| < \beta\gamma/q$. System (2) undergoes a Hopf bifurcation at (Y^*, K^*) when $\tau = \tau_j^+$ for $j = 0, 1, 2, \cdots$.*

Theorem 2.3. *Assume $\beta > q$. Let $\tau_j^\pm$ be defined in Lemma 2.1. Then we have*

(i) *the equilibrium point (Y^*, K^*) is asymptotically stable for all $\tau \geq 0$ when $k < \min\{-\sqrt{\beta^2 - q^2}/\alpha + \gamma, -\beta\gamma/q + \gamma\}$, and unstable for all $\tau \geq 0$ when $k > \max\{\sqrt{\beta^2 - q^2}/\alpha + \gamma, k_1\}$;*

(ii) *the equilibrium point (Y^*, K^*) is asymptotically stable for all $\tau \in [0, \tau_0^+)$ and unstable for all $\tau > \tau_0^+$ when $|k - \gamma| < \beta\gamma/q$. System (2) undergoes a Hopf bifurcation at (Y^*, K^*) when $\tau = \tau_j^+$ for $j = 0, 1, 2, \cdots$.*

(iii) *Assume $A^2 - 4B > 0$. The equilibrium point (Y^*, K^*) is asymptotically stable for all $\tau \in [0, \tau_0^+)$ and unstable when $\tau \in (\tau_0^+, \tau_m^-)$ where m is defined in Lemma 2.5 when $-\sqrt{\beta^2 - q^2}/\alpha + \gamma < k < -\beta\gamma/q + \gamma$. System (2) undergoes a Hopf bifurcation at (Y^*, K^*) when $\tau = \tau_j^+$ for $j = 0, 1, 2, \cdots$. When $k_1 < k < \sqrt{\beta^2 - q^2}/\alpha + \gamma$, the equilibrium point (Y^*, K^*) is unstable. System (2) undergoes a Hopf bifurcation at (Y^*, K^*) when $\tau = \tau_j^\pm$ for $j = 0, 1, 2, \cdots$.*

3 Direction and Stability of Hopf Bifurcation

From Section 2, we know that at (Y^*, K^*) the characteristic equation of linearized System (2) has a pair of purely imaginary roots $\pm i\omega_\pm$ if $\tau = \tau_j^\pm$ for each j under some conditions. Under these conditions, as the delay τ passes the critical values $\tau_j^\pm$, Hopf bifurcation occurs and periodic solutions emerge. In this section, by deriving a normal form for System (2) using a normal form theory developed by Faria and Magalhães [5, 6], we study the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions.

We first normalize the delay in System (2) by rescaling $t \to t/\tau$ to get the following system

$$\begin{cases} \frac{du_1(t)}{dt} &= \alpha\tau[(k-\gamma)u_1(t) - \beta u_2(t) + i^{(2)}u_1^2(t) + i^{(3)}u_1^3(t)] + O(|u_1|^4), \\ \frac{du_2(t)}{dt} &= \tau[ku_1(t-1) - \beta u_2(t-1) - qu_2(t) + i^{(2)}u_1^2(t-1) \\ & \quad +i^{(3)}u_1^3(t-1)] + O(|u_1|^4). \end{cases} \tag{9}$$

Let $\tau_c = \tau_j^{\pm}$ and $\tau = \tau_c + \mu$. Then μ is the bifurcation parameter for System (9) and System (9) becomes

$$\left\{\begin{array}{lcl} \frac{du_1(t)}{dt} & = & \alpha(\tau_c+\mu)[(k-\gamma)u_1(t) - \beta\tau u_2(t) + i^{(2)}u_1^2(t) + i^{(3)}u_1^3(t)] \\ & & +O(|u_1|^4), \\ \frac{du_2(t)}{dt} & = & (\tau_c+\mu)[ku_1(t-1) - \beta u_2(t-1) - qu_2(t) + i^{(2)}u_1^2(t-1) \\ & & +i^{(3)}u_1^3(t-1)] + O(|u_1|^4). \end{array}\right. \tag{10}$$

The linearization of System (10) at $(0,0)$ is

$$\left\{\begin{array}{l} \frac{du_1(t)}{dt} = \alpha\tau_c[(k-\gamma)u_1(0) - \beta u_2(0)], \\ \frac{du_2(t)}{dt} = \tau_c[ku_1(-1) - \beta u_2(-1) - qu_2(0)]. \end{array}\right. \tag{11}$$

Let

$$\eta(\theta) = A\delta(\theta) + B\delta(\theta+1)$$

where

$$A = \tau_c \begin{pmatrix} \alpha(k-\gamma) & -\alpha\beta \\ 0 & -q \end{pmatrix}, \ B = \tau_c \begin{pmatrix} 0 & 0 \\ k & -\beta \end{pmatrix}.$$

Let $C = C([-1,0], \mathbb{C}^2)$ and define a linear operator L on C as follows:

$$L\varphi = \int_{-1}^{0} d\eta(\theta)\varphi(\theta), \ \forall \varphi \in C.$$

Then System (10) can be transformed into

$$\dot{X}(t) = LX_t + F(X_t, \mu),$$

where $X = (u_1, u_2)^T, \ X_t = X(t+\theta), \ \theta \in [-1,0]$, and $F(X_t,\mu) = (F^1, F^2)^T$ where

$$\begin{array}{lcl} F^1 & = & \alpha[(k-\gamma)\mu u_1(0) - \beta\mu u_2(0) + \tau_c i^{(2)}u_1^2(0) + \tau_c i^{(3)}u_1^3(0)] + \text{h.o.t.}, \\ F^2 & = & k\mu u_1(-1) - \beta\mu u_2(-1) - q\mu u_2(0) + \tau_c i^{(2)}u_1^3(-1) + \tau_c i^{(3)}u_1^2(-1) + \text{h.o.t.}, \end{array}$$

where "h.o.t" represents high order terms. Write the Taylor expansion of F as

$$F(\varphi,\mu) = \frac{1}{2}F_2(\varphi,\mu) + \frac{1}{3!}F_3(\varphi,\mu) + \text{h.o.t.}.$$

Take the enlarged space of C

$$BC = \{\varphi : [-1,0] \to \mathbb{C}^2 : \ \varphi \text{ is continuous on } [-1,0), \ \exists \lim_{\theta\to 0^-} \varphi(\theta) \in \mathbb{C}^2\}.$$

Then the elements of BC can be expressed as $\psi = \varphi + X_0\nu$, $\varphi \in C$ and

$$X_0(\theta) = \left\{\begin{array}{l} 0, \ -1 \le \theta < 0, \\ I, \ \theta = 0, \end{array}\right.$$

where I is the identity matrix on C and the norm of BC is $|\varphi + X_0\nu| = |\varphi| + |\nu|_{\mathbb{C}^2}$. Let $C^1 = C^1([-1,0],\mathbb{C}^2)$. Then the infinitesimal generator $\mathcal{A} : C^1 \to BC$ associated with L is given by

$$\mathcal{A}\varphi = \dot{\varphi} + X_0[L\varphi - \dot{\varphi}(0)] = \begin{cases} \dot{\varphi}, & -1 \le \theta < 0, \\ A\varphi(0) + B\varphi(-1), & \theta = 0, \end{cases}$$

and its adjoint

$$\mathcal{A}^*\psi = \begin{cases} -\dot{\psi}, & 0 < s \le 1, \\ \psi(0)A + \psi(1)B, & s = 0, \end{cases} \quad \text{for } \forall \psi \in C^{1*},$$

where $C^{1*} = C^1([0,1],\mathbb{C}^{2*})$. Let $C' = C([0,1],\mathbb{C}^{2*})$ and for $\varphi \in C$ and $\psi \in C'$, define a bilinear inner product between C and C' by

$$\begin{aligned} \langle \psi, \varphi \rangle &= \psi(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi-\theta)d\eta(\theta)\varphi(\xi)d\xi \\ &= \psi(0)\varphi(0) + \int_{-1}^{0} \psi(\xi+1)B\varphi(\xi)d\xi. \end{aligned}$$

From Section 2, we know that $\pm i\tau_c\omega_0$ are eigenvalues of $\mathcal{A}$ and $\mathcal{A}^*$, where $\omega_0 = \omega_+$ or ω_-. Now we compute eigenvectors of $\mathcal{A}$ associated with $i\tau_c\omega_0$ and eigenvectors of $\mathcal{A}^*$ associated with $-i\tau_c\omega_0$. Let $q(\theta) = (\rho, k)^T e^{i\tau_c\omega_0\theta}$ be an eigenvector of $\mathcal{A}$ associated with $i\tau_c\omega_0$. Then $\mathcal{A}q(\theta) = i\tau_c\omega_0 q(\theta)$. It follows from the definition of $\mathcal{A}$ that

$$\begin{pmatrix} -\alpha(k-\gamma)\tau_c + i\tau_c\omega_0 & \alpha\beta\tau_c \\ -k\tau_c e^{-i\tau_c\omega_0} & \beta\tau_c e^{-i\tau_c\omega_0} + q\tau_c + i\tau_c\omega_0 \end{pmatrix} q(0) = 0.$$

We can obviously choose $q(\theta) = (\rho, k)^T e^{i\tau_c\omega_0\theta}$ where $\rho = \beta + (q + i\omega_0)e^{i\tau_c\omega_0}$.

Similarly, we can find an eigenvector $p(s)$ of $\mathcal{A}^*$ associated with $-i\tau_c\omega_0$

$$p(s) = \frac{1}{D}(\sigma, \alpha\beta)e^{i\tau_c\omega_0 s}, \text{ where } \sigma = -\beta e^{i\tau_c\omega_0} - q + i\omega_0$$

with D being a constant to be determined such that $\langle \bar{p}(s), q(\theta)\rangle = 1$. In fact, since

$$\langle \bar{p}(s), q(\theta)\rangle = \frac{1}{\bar{D}}[k\alpha\beta(1 + (\rho-\beta)e^{-i\tau_c\omega_0}) + \rho\bar{\sigma}]$$

we have $D = k\alpha\beta(1 + (\bar{\rho}-\beta)e^{i\tau_c\omega_0}) + \bar{\rho}\sigma$. Let P be spanned by $q, \bar{q}$ and P^* by $p, \bar{p}$. Then C can be decomposed as

$$C = P \oplus Q \text{ where } Q = \{\varphi \in C : \langle \psi, \varphi\rangle = 0, \forall \psi \in P^*\}.$$

Let $Q^1 = Q \cap C^1$. Let $\Phi(\theta) = (q(\theta), \bar{q}(\theta))$ and $\Psi(s) = \binom{\bar{p}(s)}{p(s)}$. Then $\dot{\Phi} = \Phi J$ and $\dot{\Psi} = -J\Psi$ where $J = \mathrm{diag}(i\tau_c\omega_0, -i\tau_c\omega_0)$. Define the projection $\pi : BC \to P$ by

$$\pi(\varphi + X_0\nu) = \Phi[(\Psi, \varphi) + \Psi(0)\nu].$$

Let $u = \Phi x + y$, namely

$$\begin{aligned} u_1(\theta) &= e^{i\tau_c\omega_0\theta}\rho x_1 + e^{-i\tau_c\omega_0\theta}\bar{\rho}x_2 + y_1(\theta), \\ u_2(\theta) &= e^{i\tau_c\omega_0\theta}kx_1 + e^{-i\tau_c\omega_0\theta}kx_2 + y_2(\theta). \end{aligned}$$

Then System (10) can be decomposed as

$$\begin{cases} \dot{x} = Jx + \Psi(0)F(\Phi x + y, \mu), \\ \dot{y} = A_{Q^1}y + (I - \pi)X_0F(\Phi x + y, \mu). \end{cases}$$

This can be rewritten as

$$\begin{cases} \dot{x} = Jx + \frac{1}{2}f_2^1(x, y, \mu) + \frac{1}{3!}f_3^1(x, y, \mu) + \text{h.o.t.}, \\ \dot{y} = A_{Q^1}y + \frac{1}{2}f_2^2(x, y, \mu) + \frac{1}{3!}f_3^2(x, y, \mu) + \text{h.o.t.}, \end{cases} \tag{12}$$

where

$$f_j^1(x, y, \mu) = \Psi(0)F_j(\Phi x + y, \mu), \;\; f_j^2(x, y, \mu) = (I - \pi)X_0F_j(\Phi x + y, \mu).$$

According to the normal form theory due to Faria and Magalhães [5, 6, 8], on the center manifold, System (12) can be transformed as the following normal form:

$$\dot{x} = Jx + \frac{1}{2}g_2^1(x, 0, \mu) + \frac{1}{3!}g_3^1(x, 0, \mu) + \text{h.o.t.}$$

where $g_j^1(x, 0, \mu)$ is a homogeneous polynomial of degree j in (x, μ). Let Y be a normed space and $j, p \in \mathbb{N}$. Let

$$V_j^p(Y) = \left\{ \sum_{|q|=j} c_q x^q : q \in \mathbb{N}_0^q, c_q \in Y \right\}$$

with norm $|\sum_{|q|=j} c_q x^q| = \sum_{|q|=j} |c_q|_Y$. Define M_j to be the operator in $V_j^4(\mathbb{C}^2 \times \ker \pi)$ with the range in the same space by

$$M_j(p, h) = (M_j^1 p, M_j^2 h),$$

where $(M_j^1 p)(x, \mu) = [J, p(\cdot, \mu)](x) = D_x p(x, \mu)Jx - Jp(x, \mu)$. It is easy to check that $V_j^3(\mathbb{C}^2) = \mathrm{Im}(M_j^1) \oplus \mathrm{Ker}(M_j^1)$ and

$$\mathrm{Ker}(M_j^1) = \{\mu^l x^q e_k : (q, \bar{\lambda}) = \lambda_k, k = 1, 2, q \in \mathbb{N}_0^2, |(q, l)| = j\}.$$

Hence

$$\ker(M_2^1) = \mathrm{Span}\left\{ \begin{pmatrix} \mu x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu x_2 \end{pmatrix} \right\},$$
$$\ker(M_3^1) = \mathrm{Span}\left\{ \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu^2 x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu^2 x_2 \end{pmatrix} \right\}.$$

Define

$$\tilde{f}_3^1(x,0,\mu) = f_3^1(x,0,\mu) + \frac{3}{2}[(D_x f_2^1)(x,0,\mu)U_2^1(x,\mu) + (D_y f_2^1)(x,0,\mu)U_2^2(x,\mu)]$$

where

$$U_2^1(x,\mu)|_{\mu=0} = (M_2^1)^{-1}\mathrm{Proj}_{\mathrm{Im}(M_2^1)} f_2^1(x,0,0) = (M_2^1)^{-1} f_2^1(x,0,0)$$

and $U_2^2(x,\mu)$ is determined by

$$(M_2^2 U_2^2)(x,\mu) = f_2^2(x,0,\mu).$$

Then

$$g_2^1(x,0,\mu) = \mathrm{Proj}_{\ker(M_2^1)} f_2^1(x,0,\mu),\ \ g_3^1(x,0,\mu) = \mathrm{Proj}_{\ker(M_3^1)} \tilde{f}_3^1(x,0,\mu).$$

Let us compute $g_2^1(x,0,\mu)$ first. Since

$$\frac{1}{2} f_2^1(x,0,\mu) = \begin{pmatrix} a_1\mu x_1 + a_2\mu x_2 + a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2 \\ \bar{a}_2\mu x_1 + \bar{a}_1\mu x_2 + \bar{a}_{02}x_1^2 + \bar{a}_{11}x_1x_2 + \bar{a}_{20}x_2^2 \end{pmatrix},$$

where

$$\begin{aligned}
a_1 &= -\frac{\alpha}{\bar{D}}[k\beta(q + (\beta-\rho)e^{-i\tau_c\omega_0}) + \bar{\sigma}(k(\beta-\rho) + \gamma\rho)], \\
a_2 &= -\frac{\alpha}{\bar{D}}[k\beta(q + \bar{\sigma} + \beta e^{i\tau_c\omega_0}) - \bar{\rho}(k\beta e^{i\tau_c\omega_0} + (k-\gamma)\bar{\sigma})], \\
a_{20} &= \frac{\alpha\rho^2\tau_c}{\bar{D}} i^{(2)}(e^{-2i\tau_c\omega_0}\beta + \bar{\sigma}), \\
a_{11} &= \frac{2\alpha|\rho|^2\tau_c}{\bar{D}} i^{(2)}(\beta + \bar{\sigma}), \\
a_{02} &= \frac{\alpha\bar{\rho}^2\tau_c}{\bar{D}} i^{(2)}(e^{2i\tau_c\omega_0}\beta + \bar{\sigma}),
\end{aligned} \tag{13}$$

then

$$\frac{1}{2} g_2^1(x,0,\mu) \ = \ \frac{1}{2}\mathrm{Proj}_{\ker(M_2^1)} f_2^1(x,0,\mu) = \begin{pmatrix} a_1\mu x_1 \\ \bar{a}_1\mu x_2 \end{pmatrix}.$$

Next we compute $\frac{1}{3!}g_3^1(x,0,\mu) = \frac{1}{3!}\mathrm{Proj}_{\ker(M_3^1)}\tilde{f}_3^1(x,0,\mu)$. Since the term $O(\mu^2|x|)$ is irrelevant to determine the generic Hopf bifurcation, we have

$$\begin{aligned}
\frac{1}{3!}g_3^1(x,0,\mu) &= \frac{1}{3!}\mathrm{Proj}_{\ker(M_3^1)}\tilde{f}_3^1(x,0,\mu) = \frac{1}{3!}\mathrm{Proj}_S\tilde{f}_3^1(x,0,0) + O(\mu^2|x|) \\
&= \frac{1}{3!}\mathrm{Proj}_S f_3^1(x,0,0) + \frac{1}{4}\mathrm{Proj}_S[(D_x f_2^1)(x,0,0)U_2^1(x,0) \\
&+ (D_y f_2^1)(x,0,0)U_2^2(x,0)] + O(\mu^2|x|).
\end{aligned}$$

where

$$S = \mathrm{Span}\left\{\begin{pmatrix} x_1^2x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1x_2^2 \end{pmatrix}\right\}.$$

Step 1. Compute $\frac{1}{3!}\mathrm{Proj}_{\ker(M_3^1)}f_3^1(x,0,0)$. Since

$$\frac{1}{3!}f_3^1(x,0,0) = \begin{pmatrix} a_{30}x_1^3 + a_{21}x_1^2x_2 + a_{12}x_1x_2^2 + a_{03}x_2^3 \\ \bar{a}_{03}x_1^3 + \bar{a}_{12}x_1^2x_2 + \bar{a}_{21}x_1x_2^2 + \bar{a}_{30}x_2^3 \end{pmatrix}$$

where

$$\begin{aligned}
a_{30} &= \frac{\alpha\rho^3\tau_c}{\bar{D}}[i^{(3)}(\beta e^{-3i\tau_c\omega_0} + \bar{\sigma})], \\
a_{21} &= \frac{3\alpha|\rho|^2\rho\tau_c}{\bar{D}}[i^{(3)}(\beta e^{-i\tau_c\omega_0} + \bar{\sigma})], \\
a_{12} &= \frac{3\alpha|\rho|^2\bar{\rho}\tau_c}{\bar{D}}[i^{(3)}(\beta e^{i\tau_c\omega_0} + \bar{\sigma})], \\
a_{03} &= \frac{\alpha\bar{\rho}^3\tau_c}{\bar{D}}[i^{(3)}(\beta e^{3i\tau_c\omega_0} + \bar{\sigma})],
\end{aligned}$$

we have

$$\frac{1}{3!}\mathrm{Proj}_{\ker(M_3^1)}f_3^1(x,0,0) = \begin{pmatrix} a_{21}x_1^2x_2 \\ \bar{a}_{21}x_1x_2^2 \end{pmatrix}.$$

Step 2. Compute $\frac{1}{2}\mathrm{Proj}_S[D_x f_2^1(x,0,0)U_2^1(x,0)]$. The elements of the canonical basis of $V_2^2(\mathbb{C}^2)$ are

$$\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu^2 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu x_1 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu^2 \end{pmatrix},$$

whose images under $\frac{1}{i\omega_0}M_2^1$ are, respectively

$$\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, -\begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}, -3\begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, -2\begin{pmatrix} \mu x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu^2 \\ 0 \end{pmatrix},$$
$$3\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix}, -\begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}, 2\begin{pmatrix} 0 \\ \mu x_1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu^2 \end{pmatrix}.$$

Hence

$$U_2^1(x,0)=\frac{1}{i\omega_0}\begin{pmatrix}a_{20}x_1^2-a_{11}x_1x_2-\frac{1}{3}a_{02}x_2^2\\ \frac{1}{3}\bar{a}_{02}x_1^2+\bar{a}_{11}x_1x_2-\bar{a}_{20}x_2^2\end{pmatrix},$$

and

$$\frac{1}{2}\mathrm{Proj}_S[D_xf_2^1(x,0,0)U_2^1(x,0)]=\begin{pmatrix}C_1x_1^2x_2\\ \bar{C}_1x_1x_2^2\end{pmatrix}$$

where

$$\begin{aligned}C_1 &= \frac{2}{i\omega_0}(2|a_{02}|^2-3a_{20}a_{11}+3|a_{11}|^2)\\ &= -\frac{i\alpha^2|\rho|^2\rho\tau_c^2(i^{(2)})^2}{12\bar{D}|D|^2}[6\bar{D}\bar{\rho}|\beta+\sigma|^2-3D\rho(\beta+\bar{\sigma})(\beta e^{-2i\tau_c\omega_0}+\bar{\sigma})\\ &+ \bar{D}\bar{\rho}|\beta e^{-2i\tau_c\omega_0}+\sigma|^2].\end{aligned}$$

Step 3. Compute $\frac{1}{2}\mathrm{Proj}_S[(D_yf_2^1)(x,0,0)U_2^2(x,0)]$, where $U_2^2(x,0)$ is a second-order homogeneous polynomial in (μ,x_1,x_2) with coefficients in Q^1. Let

$$h(x)(\theta)=U_2^2(x,0)=h_{20}(\theta)x_1^2+h_{11}(\theta)x_1x_2+h_{02}(\theta)x_2^2.$$

The coefficients $h_{jk}=(h_{jk}^1,h_{jk}^2)^T$ are determined by $M_2^2h(x)=f_2^2(x,0,0)$ or

$$D_xh(x)Bx-A_{Q^1}(h(x))=(I-\pi)X_0F_2(\Phi x,0)$$

which is equivalent to

$$\begin{gathered}\dot{h}(x)-D_xh(x)Bx=\Phi\Psi(0)F_2(\Phi x,0),\\ \dot{h}(x)(0)-Lh(x)=F_2(\Phi x,0),\end{gathered}$$

where $\dot{h}$ denotes the derivative of $h(x)(\theta)$ with respect to θ. Note that

$$F_2(\Phi x,0)=A_{20}x_1^2+A_{11}x_1x_2+A_{02}x_2^2$$

where

$$\begin{aligned}A_{20} &= (2i^{(2)}\alpha\rho^2\tau_c,2i^{(2)}\rho^2\tau_ce^{-2i\tau_c\omega_0})^T,\\ A_{11} &= (4i^{(2)}\alpha|\rho|^2\tau_c,2i^{(2)}\alpha|\rho|^2\tau_c)^T,\\ A_{02} &= (2i^{(2)}\alpha\bar{\rho}^2\tau_c,2i^{(2)}\bar{\rho}^2\tau_ce^{2i\tau_c\omega_0})^T.\end{aligned}$$

Comparing the coefficients of x_1^2,x_1x_2,x_2^2 of these equations, it is not hard to verify that $\bar{h}_{02}=h_{20},\bar{h}_{11}=h_{11}$ and that h_{20},h_{11} satisfy the following equations

$$\begin{cases}\dot{h}_{20}-2i\tau_c\omega_0h_{20}=\Phi\Psi(0)A_{20},\\ \dot{h}_{20}(0)-Lh_{20}=A_{20},\end{cases}\tag{14}$$

and

$$\begin{cases} \dot{h}_{11} = \Phi\Psi(0)A_{11}, \\ \dot{h}_{11}(0) - Lh_{11} = A_{11}. \end{cases} \tag{15}$$

Noting that $f_2^1(x,0,0) = \Psi F_2(\Phi x, 0)$, we deduce

$$\frac{1}{2}(D_y f_2^1)h(x,0,0) = \begin{pmatrix} \frac{\alpha\tau_c i^{(2)}}{2\bar{D}}[\beta(\rho e^{-i\tau_c\omega_0}x_1 + \bar{\rho}e^{i\tau_c\omega_o}x_2)h^1(-1) + \bar{\sigma}(\rho x_1 + \bar{\rho}x_2)h^1(0)] \\ \frac{\alpha\tau_c i^{(2)}}{2D}[\beta(\rho e^{-i\tau_c\omega_0}x_1 + \bar{\rho}e^{i\tau_c\omega_o}x_2)h^1(-1) + \bar{\sigma}(\rho x_1 + \bar{\rho}x_2)h^1(0)] \end{pmatrix}$$

where

$$h^1(-1) = h_{20}^1(-1)x_1^2 + h_{11}^1(-1)x_1x_2 + h_{02}^1(-1)x_2,$$
$$h^1(0) = h_{20}^1(0)x_1^2 + h_{11}^1(0)x_1x_2 + h_{02}^1(0)x_2^2.$$

and hence

$$\frac{1}{2}\mathrm{Proj}_S[(D_y f_2^1)h](x,0,0) = \begin{pmatrix} C_2 x_1^2 x_2 \\ \bar{C}_2 x_1 x_2^2 \end{pmatrix},$$

where

$$C_2 = \frac{\alpha\tau_c i^{(2)}}{\bar{D}}[e^{-i\omega_0\tau_0}\beta\rho h_{11}^1(-1) + \rho\bar{\sigma}h_{11}^1(0) + e^{i\omega_0\tau_0}\beta\bar{\rho}h_{20}^1(-1) + \bar{\rho}\bar{\sigma}h_{20}^1(0)].$$

Here h_{20}, h_{11} are determined by System (14) and System (15). After long but basic calculations, we obtain

$$\begin{aligned}
&h_{20}^1(0) = \\
&(2i^{(2)}\alpha\rho^2 e^{-3\tau_c\omega_0}(\bar{D}(-ie^{2i\tau_c\omega_0}(2De^{3i\tau_c\omega_0}\sigma)\omega(-iq+2\omega_0) \\
&+k\alpha\beta(\beta + e^{2i\tau_c\omega_0}\sigma)((-1+e^{2i\tau_c\omega_0})\beta - 2ie^{i\tau_c\omega_0}\omega_0)) \\
&+e^{i\tau_c\omega_0}(\beta + e^{2i\tau_c\omega_0}\sigma)(-ie^{i\tau_c\omega_0}k\alpha\beta + ie^{3i\tau_c\omega_0}k\alpha\beta \\
&+2\beta\omega_0 + 2e^{2i\tau_c\omega_0}(q+2i\omega_0)\omega_0)\bar{\rho}) + D(2e^{i\tau_c\omega_0}\rho(\beta + e^{2i\tau_c\omega_0}(q+2i\omega_0))\omega_0 \\
&-ik\alpha\beta((-1+e^{2i\tau_c\omega_0})\beta + \rho - 2e^{2i\tau_c\omega_0}\rho - 2ie^{3i\tau_c\omega_0}\omega_0))(\beta + e^{i\tau_c\omega_0}\bar{\sigma})) \\
&/(\omega_0 D(-i\beta + e^{2i\tau_c\omega_0}(-iq+2\omega_0)) - \alpha(\beta\gamma - ie^{2i\tau_c\omega_0}(k-\gamma)(-iq+2\omega_0))\bar{D}),
\end{aligned}$$

$$\begin{aligned}
&h_{20}^1(-1) = \\
&\frac{2i^{(2)}\alpha\rho^2 e^{-5\tau_c\omega_0}}{\omega_0|D|^2}[i(-1+e^{2i\tau_c\omega_0})(e^{2i\tau_c\omega_0}(\beta + e^{2i\tau_c\omega_0}\sigma)\bar{D}\bar{\rho} + D\rho(\beta + e^{2i\tau_c\omega_0}\bar{\sigma})) \\
&+(\bar{D}(-ie^{2i\tau_c\omega_0}(2De^{2i\tau_c\omega_0}\omega_0(-iq+2\omega) + k\alpha\beta(\beta + e^{2i\tau_c\omega_0}\sigma)((-1+e^{i\tau_c\omega_0})\beta \\
&-2ie^{2i\tau_c\omega_0}\omega_0)) + e^{i\tau_c\omega_0}(\beta + e^{2i\tau_c\omega_0})(-ie^{i\tau_c\omega_0}k\alpha\beta + ie^{3i\tau_c\omega_0}k\alpha\beta + 2\beta\omega_0 \\
&+2e^{2i\tau_c\omega_0}(q+2i\omega_0)\omega_0)\bar{\rho}) + D(2e^{2i\tau_c\omega_0}\rho(\beta + e^{2i\tau_c\omega_0}(q+2i\omega_0))\omega_0 \\
&-ik\alpha\beta((-1+e^{2i\tau_c\omega_0})\beta + \rho - e^{2i\tau_c\omega_0} - 2ie^{3i\tau_c\omega_0}))(\beta + e^{2i\tau_c\omega_0}\bar{\sigma}))] \\
&/(2\omega_0(-i\beta + e^{2i\tau_c\omega_0}(-iq+2\omega)) - \alpha(\beta\gamma - ie^{2i\tau_c\omega_0}(k-\gamma)(-iq+2\omega_0)))
\end{aligned}$$

and

$$
\begin{aligned}
&h^1_{11}(0) = \\
&\frac{4i^{(2)}|\rho|^2 e^{-i\tau_c\omega_0}}{(\beta\gamma+(-k+\gamma)q)|D|^2}[e^{i\tau_c\omega_0}\bar{D}(-Dq+k\alpha\beta(\beta+\sigma)(-1+e^{i\tau_c\omega_0}\beta\tau_c) \\
&+(\beta+\sigma)(-q+\beta(-1+e^{i\tau_c\omega_0}k\alpha\tau_c))\bar{\rho}) - D(e^{i\tau_c\omega_0}(k\alpha\beta-(\beta+q)\rho) \\
&+k\alpha\beta(-\beta+\rho)\tau_c)(\beta+\bar{\sigma})], \\
&h^1_{11}(-1) \\
&\frac{4i^{(2)}|\rho|^2 e^{-i\tau_c\omega_0}}{|D|^2}[-e^{-2i\tau_c\omega_0}\alpha\tau_c((\beta+\sigma)\bar{D}\bar{\rho}+De^{-2i\tau_c\omega_0}\rho(\beta+\bar{\sigma})) \\
&-\frac{1}{\beta\gamma+(-k+\gamma)q}(\bar{D}e^{i\tau_c\omega_0}(-Dq+k\alpha\beta(\beta+\sigma)(-1+e^{i\tau_c\omega_0}\beta\tau_c) \\
&-(\beta+\sigma)(-q+\beta(-1+e^{i\tau_c\omega_0}k\alpha\tau_c))\bar{\rho}) - D(e^{i\tau_c\omega_0}(k\alpha\beta-(\beta+q)\rho) \\
&+k\alpha\beta(-\beta+\rho)\tau_c)(\beta+\bar{\sigma}))].
\end{aligned}
$$

Collecting the results above, we obtain

$$\frac{1}{3!}g^1_3(x,0,\mu) \quad = \quad \begin{pmatrix} b_{21}x_1^2x_2 \\ \bar{b}_{21}x_1x_2^2 \end{pmatrix} + O(\mu^2|x|),$$

where $b_{21} = a_{21} + \frac{1}{2}(C_1 + C_2)$. Therefore, System (10) can be transformed into the following normal form:

$$\begin{cases} \dot{x}_1 = i\tau_c\omega_0 x_1 + a_1\mu x_1 + b_{21}x_1^2x_2 + \text{h.o.t.}, \\ \dot{x}_2 = -i\tau_c\omega_0 x_2 + \bar{a}_1\mu x_2 + \bar{b}_{21}x_1x_2^2 + \text{h.o.t.}, \end{cases} \tag{16}$$

where a_1 is given in (13). Let $x_1 = w_1 + iw_2, x_2 = w_1 - iw_2$ and $w_1 = r\cos\xi, w_2 = r\sin\xi$. Then (16) can be further written as

$$\begin{cases} \dot{r} = a\mu r + br^3 + \text{h.o.t.}, \\ \dot{\xi} = \tau_c\omega_0 + \text{h.o.t.}, \end{cases}$$

where $a = \text{Re}[a_1]$ and $b = \text{Re}[b_{21}]$. Hence the first Lyapunov coefficient is $l_1(\mu) = b + O(\mu)$, see [3, 13].

Theorem 3.1. *Let a and b be given above.*

(i) *The bifurcating periodic solution is stable if $b < 0$, and unstable if $b > 0$;*

(ii) *The Hopf bifurcation is supercritical if $ab < 0$, and subcritical if $ab > 0$.*

Remark. *The coefficient a is given by*

$$
\begin{aligned}
&a = \mathrm{Re}[a_1] = \\
&\frac{k^2\alpha^2(q^2+\omega_0^2)}{\beta^2(\alpha^2\gamma^2+\omega_0^2)|D|^2}[-\alpha\omega_0^2(\beta^2-q^2-\omega_0^2)(\beta^2\gamma+(k-\gamma)(q^2+\omega_0^2)) \\
&+\alpha^3(-\beta^4\gamma^3+\beta^2\gamma(k^2-3k\gamma+2\gamma^2)(q^2+\omega_0^2)+(k-\gamma)^3(q^2+\omega_0^2)^2 \\
&+\omega_0^4(\beta^2 q-(q^2+\omega_0^2)(q-\omega_0^2))-\alpha^4(k-\gamma)^2((k-\gamma)^2 q(1+q\tau_c)(q^2+\omega_0^2) \\
&-\beta^2\gamma^2(q+q^2\tau_c+\tau_c\omega_0^2))+\alpha^2\omega_0^2(2k\gamma(-\beta^2 q+2q^3+q^4\tau_c+2q\omega_0^2-\tau_c\omega_0^4) \\
&+k^2(\beta^2 q-2q^3-q^4\tau_c-2q\omega^2+\tau_c\omega_0^4)+\gamma^2(-(q^2+\omega_0^2)(2q+q^2\tau_c-\tau_c\omega_0^2) \\
&+\beta^2(2q+q^2\tau_c+\tau_c\omega_0^2))))].
\end{aligned}
$$

Although the explicit algorithm is derived to compute b, it is difficult to determine the sign of b for general $\alpha, \beta, \gamma, k, q$. But if $i^{(2)} = 0$, it is easy to see $C_1 = C_2 = 0$ and hence b can be simply expressed as

$$
\begin{aligned}
b \;=\; & -\frac{3i^{(3)}}{|D|^2}(\beta^2+q^2+\omega_0^2+2\beta q\cos(\tau_c\omega_0)-2\beta\omega\sin(\tau_c\omega_0))(-k\alpha\beta^2 q+2\beta^2 q \\
&+2\beta^2 q^2+q^4-k\alpha\beta^2 q^2\tau_c+2\beta^2\omega_0^2+2q^2\omega^2-k\alpha^2\beta^2\tau_c\omega_0^2+\omega_0^4 \\
&+\beta(\beta^2 q+3q(q^2+\omega_0^2)-k\alpha(q^2+q^3\tau_c-\omega_0^2+q\tau_c\omega_0^2))\cos(\tau_c\omega_0) \\
&+\beta^2(q^2-\omega_0^2)\cos(2\tau_c\omega_0)-\beta^3\omega_0\sin(\tau_c\omega_0)+2k\alpha\beta q\sin(\tau_c\omega_0) \\
&-3\beta q^2\omega_0\sin(\tau_c\omega_0) \\
&+k\alpha\beta q^2\tau_c\omega_0\sin(\tau_c\omega_0)-3\beta\omega_0^3\sin(\tau_c\omega_0)-2\beta^3 q\omega_0\sin(2\tau_c\omega_0)).
\end{aligned}
$$

4 Numerical Simulations

In this section, we give some examples to illustrate the theoretical results obtained in the previous sections.

Example 1. Let $\alpha = 1,\ \beta = 0.8,\ \gamma = 0.5625,\ q = 0.9$ and

$$I(s) = \tanh(0.5s).$$

Then $(0, 0)$ is an equilibrium point of System (2), $k = 0.5$, $i^{(2)} = 0$, $i^{(3)} = -0.041667$. Hence $\omega_+ = 0.6066$, and $\tau_0^+ = 3.1382$. Take $\tau = 2.5$. According to Theorem 2.2 (ii), the trivial equilibrium point $(0, 0)$ is asymptotically stable, (Figure 1).

Example 2. Let $\alpha = 1,\ \beta = 0.8,\ \gamma = 0.5625,\ q = 0.9$ and

$$I(s) = \tanh(0.5s).$$

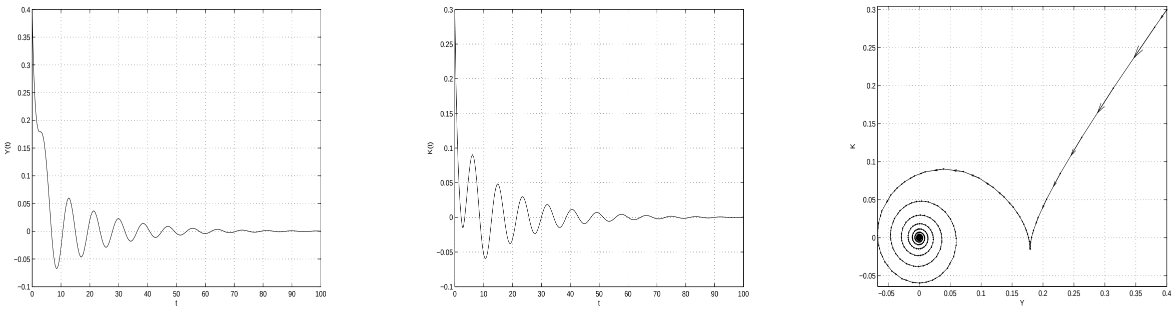

Figure 1: The equilibrium point $(0,0)$ is asymptotically stable when $\tau < \tau_0^+$.

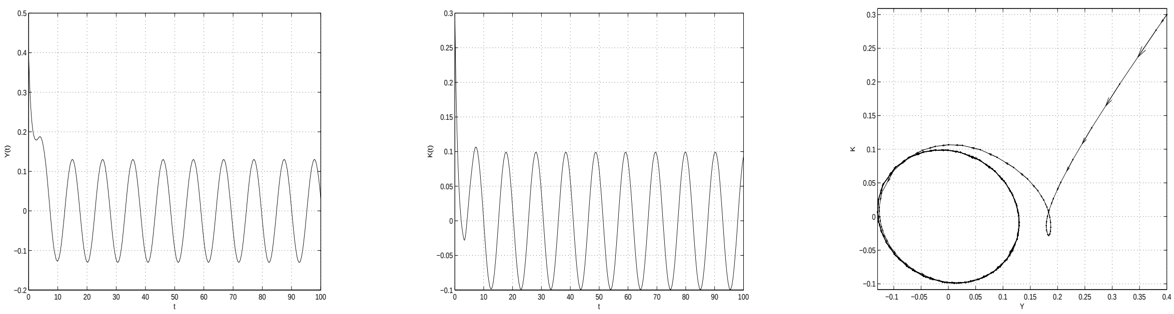

Figure 2: The stable periodic orbit generated by Hopf bifurcation when $\beta < q$.

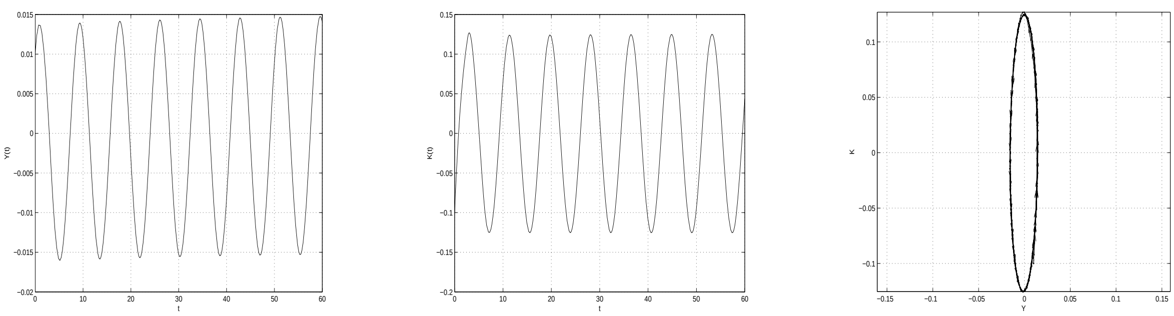

Figure 3: The stable periodic orbit generated by Hopf bifurcation when $\beta > q$.

Then $k = 0.5,\ i^{(2)} = 0,\ i^{(3)} = -0.041667$ and hence $\omega_+ = 0.6066, \tau_0^+ = 3.1382$. Take $\tau_c = \tau_0^+,\ \mu = 0.001$. After using the algorithm in Section 3, we have

$$a = 2.0772,\ b = -0.0362,$$

and hence the bifurcating periodic solution is stable and the Hopf bifurcation is supercritical (Figure 2).

Example 3. Let $\alpha = 0.1,\ \beta = 0.9,\ \gamma = 0.5625,\ q = 0.5$ and

$$I(s) = \tanh(0.9s).$$

Then $k = 0.9,\ i^{(2)} = 0,\ i^{(3)} = -0.243$ and hence $\omega_+ = 0.7503,\ \tau_0^+ = 2.7185$. Take $\tau_c = \tau_0^+,\ \mu = 0.001$. After using the algorithm in Section 3, we have

$$a = 1.2365,\ b = -0.0002,$$

and hence the bifurcating periodic solution is stable and the Hopf bifurcation is supercritical (Figure 3).

References

[1] A. Agliari, R. Dieci and L. Gardini, Homoclinic tangles in a Kaldor-like business cycle model, *J. Econ. Behav. Organ.* **62** (2007), 324-347.

[2] G. I. Bisichi, R. Dieci, G. Rodano and E. Saltari, Multiple attractors and global bifurcations in a Kaldor-type business cycle model, *J. Evol. Econ.* **11** (2001), 527-544.

[3] S. N. Chow, C. Li and D. Wang, *Normal Forms and Bifurcation of Planar Vector Fields*, Cambridge, New York, 2004.

[4] W. W. Chang and D. J. Smith, The existence and persistence of cycles in a nonlinear model: Kaldor's 1940 model re-examined, *Rev. Econ. Stud.* **38** (1971), 37-44.

[5] T. Faria and L. T. Magalhães, Normal forms for retarded functional differential equations and applications to Hopf bifurcation, *J. Differential Equations* **122** (1995), 181-200.

[6] T. Faria and L. T. Magalhães, Normal forms for retarded functional differential equations and applications to Bogdanov-Takens singularity, *J. Differential Equations* **122** (1995), 201-224.

[7] J. Grasman and J. J. Wentzel, Co-existence of a limit cycle and an equilibrium in Kaldor's business cycle model and its consequences, *J. Econ. Behav. Organ.* **24** (1994), 369-377.

[8] J. Hale, L. Magalhães, W. Oliva, *Dynamics in Infinite Dimensions*, Springer, New York, 2002.

[9] A. Kaddar and H. Talibi Alaoui, Hopf bifurcation analysis in a delayed Kaldor-Kalecki model of business cycle, *Nonlinear Anal.: Modelling and Control* **13** (2008), 439-449.

[10] N. Kaldor, A model of the trade cycle, *Economic Journal* **50** (1940), 78-92.

[11] N. Kalecki, A macrodynamic theory of business cycles, *Ecconometrica* **3** (1935), 327-344.

[12] N. Kalecki, A theory of the business cycle, *Rev. Stud.* **4** (1937), 77-97.

[13] Yu. Kuznetsov, *Elements of Applied Bifurcation Theory*, Springer, New York, 2004.

[14] A. Krawiec and M. Szydlowski, The Kaldor-Kalecki business cycle model, *Annals of Operations Research* **89** (1999), 89-100.

[15] A. Krawiec and M. Szydlowski, The Kaldor-Kalecki model of business cycle, *J. of Nonlinear Math. Phys. Supp. Proceedings NEED 99*, **8** (2001), 266-271.

[16] A. Krawiec and M. Szydlowski, The Hopf bifurcation in the Kaldor-Kalecki model, in: *Computation in Economics, Finance and engineering: Economics Systems, S. Holly and S. Greenblatt (Eds.)* Elsevier, 2000, 391-396.

[17] A. Krawiec and M. Szydlowski, On nonlinear mechanics of business cycle model, *Regul. Chaotic Dyn.* **6** (2001), 101-118.

[18] S. Ruan and J. Wei, On the zeros of transcendental functions with applications to stability of delay differential equations with two delays, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **10** (2003), 863-874.

[19] M. Szydlowski and A. Krawiec, The stability problem in the Kaldor-Kalecki business cycle model, *Chaos, Solitons and Fract.* **25** (2005), 299-305.

[20] M. Szydlowski, A. Krawiec and J. Tobola, Nonlinear oscillations in business cycle model with time lags, *Chaos, Solitons and Fract.* **12** (2001), 505-517.

[21] Y. Takeucki and T. Yamamura, The stability analysis of the Kaldor with time delays: monetary policy and government budget constraint, *Nonlinear Anal.: Real World Applications* **5** (2004), 277-308.

[22] X. P. Wu and L. Wang, Multi-parameter bifurcations of the Kaldor-Kalecki model of business cycle with delay, *Nonlinear Anal.: Real World Applications*, to appear.

[23] H. R. Varian, Catastrophe theory and the business cycle, *Econ. Inq.* **17** (1979), 14-28.

[24] C. Zhang and J. Wei, Stability and bifurcation analysis in a kind of business cycle model with delay, *Chaos, Solitons and Fract.* **22** (2004), 883-896.